Problem Set III: due Monday, February 26

1) a) Derive the Hasegawa-Wakatani equations. Do this like the derivation for reduced MHD, but:

   i) neglect inductive effects, and all magnetic perturbations
   ii) retain electron pressure in the Ohm's Law
   iii) take electrons isothermal
   iv) derive an equation for electron density, including parallel electron flow.

   b) Discuss the conservation properties for this system.

   c) Derive the quasi-linear equations for the H-W system. What do they mean?

   d) Derive the mean vorticity and particle flux.

   f) Relate the vorticity flux to the Reynolds stress.

2) a) Starting from the H-W equations, derive the Hasegawa-Mima equation in the limit $k_\parallel^2 \mathrm{v}_{The}^2 / \omega\nu \to \infty$. What is the physics of this limit?

   b) What quantities are conserved by the Hasegawa-Mima Equation?

   c) What are the linear waves of the H-M system? Obtain the dispersion relation.

   d) Recover these in the H-W system, for $k_\parallel^2 \mathrm{v}_{Th}^2 / \omega\nu > 1$ but not infinite. Discuss your result. How does instability occur?

3) This problem asks you to explore the Current Convective Instability (CCI) in a homogeneous medium and its sheared field relative, the Rippling Instability.

a) Consider first a current carrying plasma in a straight magnetic field $\underline{B} = B_0 \hat{z}$ - i.e. ignore the poloidal field, etc. Noting that the resistivity η is a function of temperature (ala' Spitzer - c.f. Kulsrud 8.7), calculate the electrostatic resistive instability growth rate, assuming T evolves according to:

$$\frac{\partial T}{\partial t} + \underline{v} \cdot \underline{\nabla} T - \chi_{\parallel} \partial_z^2 T - \chi_{\perp} \nabla_{\perp}^2 T = 0$$

and the electrostatic Ohm's Law is just

$$-\partial_z \phi = \frac{1}{\eta} \frac{d\eta}{dT} \tilde{T} J_0 .$$

b) *Thoroughly* discuss the physics of this simple instability, i.e.

- what is the free energy source?
- what is the mechanism?
- what are the dampings and how do they restrict the unstable spectrum?
- how does spectral asymmetry enter?
- what is the cell structure?

c) Use quasilinear theory and the wave breaking limit to estimate the heat flux from the C.C.I.

d) Now, consider the instability in a *sheared* magnetic field.

i.) What difficulties enter the analysis?

ii.) Resolve the difficulty by considering coupled evolution of vorticity, Ohm's Law (in electrostatic limit but with temperature fluctuations) and electron temperature. Compute the growth rate in the limit $\chi_{\parallel}$, $\chi_{\perp} \rightarrow 0$. Compute the mode width. Discuss how asymmetry enters here. Explain why.

e) Noting that $\chi_{\parallel} >> \chi_{\perp}$ (why? - see Kulsrud 8.7), estimate when parallel thermal conduction becomes an important damping effect. Can $\chi_{\parallel}$ alone ever absolutely stabilize the rippling mode?

f) Calculate the quasilinear heat flux and use the breaking limit to estimate its magnitude.

4) *Taylor in Flatland*

Taylor awakes one morning, and finds himself in Flatland, a 2D world. Seeking to relax, he sets about reformulating his theory for that planar universe.

a) Write down the visco-resistive 2D MHD equations, and show that *three* quadratic quantities are conserved, as $\eta \to 0$, $\nu \to 0$.

b) Which of these is the most likely to constrain magnetic relaxation? Argue that

i.) the local version of this quantity is conserved for an 'flux circle', as $\eta \to 0$,
ii.) the global version is the most "rugged", for finite η.

c) Formulate a 2D Taylor Hypothesis - i.e. that magnetic energy is minimized while the quantity you identified from b.) ii.) is conserved. What equation describes this state? Show that the solution is force-free. What quantity is constant in Flatland? Hence, what is the endstate of Taylor relaxation in 2D?

d) Consider the possibility that $\nu >> \eta$ in Flatland. Derive the mean field evolution equation for mean magnetic potential. Discuss!

e) *Optional - Extra Credit* - Describe the visit of the Terrifying Torus to Flatland. How would 2D Taylor perceive this apparition?

N.B. You may find it useful to consult *Flatland*, by E. Abbott.

5) Kulsrud; Chapter 7, Problem 4

6) Kulsrud; Chapter 7, Problem 2

7) Kulsrud; Chapter 11, Problem 1. Ignore the last paragraph.

8) Reformulate the Sweet–Parker Reconnection problem for weak collisionality. Assume a uniform, strong guide field $B_o\hat{z}$ orthogonal to the plane of reconnection. What can be said about the reconnection speed? [Note: This is an open-ended problem that asks you to synthesize the stories of the current-driven ion–acoustic instability and the resulting scattering of momentum with the S–P problem. You may find it useful to consult relevant parts of Kulsrud, Chapter 14.]

9) a) Derive the DNLS for weakly compressible nonlinear Alfven packets, both by heuristic and by iterative methods. For the latter, consider the fast and slow evolution of $\tilde{B}$ using the induction equation.

b) Consider how the DNLS might change in long ion mean free path regimes. What types of dissipation might enter, and affect the structure of the wave packet equation? [N.B. This is open-ended.]

c) How might one treat the case where beta → 1 ?

d) Derive a weak turbulence analogue of the Alfvenic packet steepening story by combining:

— the effects of Alfven wave radiation pressure on a parallel acoustic wave
— wave kinetics for the Alfvenic packet

Focus on the effects of refraction and note the analysis of weak Langmuir turbulence.